Models for Flicker Noise in DSN Oscillators

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A mathematically tractable model for flicker noise is presented. The model is not stationary, but has stationary increments. It behaves like flicker noise when subjected either to high-pass filtering or to direct spectral measurements. Effects of a detrending operation on these measurements are investigated. The model is expressed as a limit of stationary processes. The model of Barnes and Allan is reviewed, and the performances of the two models are compared.

I. Heuristic Description of the Model

The paradox of flicker noise appears in the study of certain types of time series, which include oscillator frequency fluctuations and noise in semiconductors. Experiments that attempt to measure the spectral density of such time series at low angular Fourier frequencies ω have yielded densities of the order $1/|\omega|$ or even $1/|\omega|^\alpha$, $\alpha > 1$ (Refs. 1 and 2). No rolloff from this behavior has been observed down to frequencies of the order 1 cycle per year. Since $1/|\omega|^\alpha$, $\alpha > 1$, is not integrable over the low-frequency range, there is no stationary process with such a spectral density.

This situation has been dealt with by two approaches (Refs. 3–8):

(1) Assume that a stationary formalism can still be used. Plug $1/|\omega|$ into formulas as if it were the spectral density of a stationary process. If the resulting integrals converge, those results are meaningful. An objection to this approach is that it is mathematically unsound.

(2) Assume that the $1/|\omega|$ behavior cuts off in some way below a frequency ϵ lower than any frequency of interest. Then use a mathematically sound stationary formalism and obtain results depending perhaps on ϵ. Investigate what happens as $\epsilon \to 0$. An objection here is that one has to assume the existence of something never observed, namely, an artificially imposed cutoff.

We believe that there really is no cutoff. Hence we must look to nonstationary processes to find a sound mathematical model for the phenomenon. This is not to say that we abandon the idea that flicker noise is produced by a stationary mechanism; after all, an ordinary random walk is a nonstationary process, but consists of the partial sums of a stationary sequence. This idea is the germ of our model. It is often said that when you pass a stationary process $X(t)$ with spectral density $f(\omega)$ through a perfect integrator, you get a process $Y(t)$ with spectral density $f(\omega)/\omega^2$. However, such a Y is not stationary, but has stationary increments, i.e., the processes $Y_\delta(t) = Y(t + \delta) - Y(t)$ are stationary. We will not say that Y has spectral density $f(\omega)/\omega^2$, but will show that $f(\omega)/\omega^2$ can be associated with Y in an experimentally meaningful way.

To make a model for flicker noise, define $f(\omega) = |\omega|$, at least for small ω. The mathematics of the model will require a high-frequency rolloff from the $1/|\omega|$ behavior. Accordingly, our model is as follows: Send white noise through a high-pass filter that attenuates low frequencies at a rate of 3 dB per octave. This gives a stationary process $X(t)$ with spectral density $|\omega|$ for small ω. Then our model $Y(t)$ for flicker noise is given by

$$Y(t) = \int_0^t X(s)ds \tag{1}$$

We will also add a constant term and a linear drift term, since such dc components are natural to processes with stationary increments. Besides, oscillators often exhibit linear frequency drift, with random fluctuations superimposed.

We will not try to "prove" that the process Y has a spectral density $1/|\omega|$. There is no such thing. We will simply subject Y to the same measurements on paper that have been performed on flicker noise in the laboratory. These include (1) passages through high-pass filters, and (2) attempts to measure spectral density directly. If Y behaves like flicker noise, then to this extent our model is successful. We can then entertain hopes of finding a physical mechanism that generates the model.

Fortunately, the calculations are easy to carry out, for processes with stationary increments are mathematically tractable. They are an immediate generalization of stationary processes, and their spectral theory is almost as simple (Refs. 9, 10, and 11). On the way, we will point out how the two approaches fit into the picture.

II. Stationary Processes and Processes with Stationary Increments

We will be concerned here only with first and second moment properties of processes. Accordingly, "stationary" means "weakly stationary," i.e., that the covariance depends only on time differences.

If $X(t)$, $-\infty < t < \infty$, is a continuous-time, complex-valued, mean-continuous, stationary process with spectral distribution function F, then

$$X(t) = \int_{-\infty}^{\infty} \exp(i\omega t) dZ(\omega) \tag{2}$$

where Z is a process with orthogonal increments such that $E|dZ(\omega)|^2 = dF(\omega)$. The function F is increasing and bounded on $(-\infty, \infty)$. If X has a spectral density f, then $dF(\omega) = f(\omega)d\omega$.

A stochastic integral

$$\int_{-\infty}^{\infty} \phi(\omega)\, dZ(\omega)$$

of which Eq. (2) is an example, is defined for functions ϕ such that

$$\int_{-\infty}^{\infty} |\phi(\omega)|^2 dF(\omega) < \infty \tag{3}$$

The main property of this integral is

$$E\left(\int_{-\infty}^{\infty} \phi dZ\right)\left(\overline{\int_{-\infty}^{\infty} \psi dZ}\right) = \int_{-\infty}^{\infty} \phi(\omega)\overline{\psi(\omega)}\, dF(\omega) \tag{4}$$

if ϕ and ψ satisfy Eq. (3).

The theory of stationary processes can be found in Refs. 9–14. A process $Y(t)$, $-\infty < t < \infty$, is said to have stationary increments if $E[Y(s) - Y(t)] = a(s - t)$ for some number a, and if

$$E[Y(t + \tau_1) - Y(t)][\overline{Y(t + \tau_2) - Y(t)}]$$

does not depend on t. If Y is mean-square differentiable, then its derivative Y' is stationary, mean-continuous, and satisfies

$$Y(t) - Y(0) = \int_0^t Y'(s)\, ds$$

the integral being in the mean-square sense. If Y' is the process X of Eq. (2), then

$$\begin{aligned} Y(t) - Y(0) &= \int_0^t ds \int_{-\infty}^{\infty} dZ(\omega) \exp(i\omega s) \\ &= \int_{-\infty}^{\infty} dZ(\omega) \int_0^t ds \exp(i\omega s) \end{aligned}$$

$$Y(t) = Y(0) + \int_{-\infty}^{\infty} \frac{\exp(i\omega t) - 1}{i\omega} dZ(\omega) \tag{5}$$

The interchange of orders of integration leading to Eq. (5) may be carried out because $\int dF < \infty$; see Rozanov's book (Ref. 14, p. 12) for the relevant theorem.

By considering the stationary processes $Y(t+\delta) - Y(t)$, it can be shown that any mean-continuous process with stationary increments, differentiable or not, has a representation of form (5), where the increasing function F that corresponds to Z via $E\,|\,dZ\,|^2 = dF$ no longer need be bounded, but merely satisfies

$$\int_{-\infty}^{\infty} \frac{dF(\omega)}{1+\omega^2} < \infty \tag{6}$$

This is equivalent to Eq. (3), where $\phi(\omega) = [\exp(i\omega t) - 1]/i\omega$. For example, if $F(\omega) = \omega/(2\pi)$, then $Y(t) - Y(0)$ has the same covariances as Brownian motion. This may be shown by using Eqs. (5) and (4).

More detailed accounts of these processes may be found in Ref. 11, p. 86 ff., and in Ref. 10.

We will concentrate our attention on the case in which F has a jump at 0, but elsewhere has a density f. Then Z has a jump Z_0 at 0, orthogonal to $dZ(\omega)$ when $\omega \neq 0$. Since $[\exp(i\omega t) - 1]/i\omega$ is considered as having the value t when $\omega = 0$, the contribution of Z_0 to Eq. (5) is $Z_0 t$. Indeed, the presence of linear drift is part of the nature of processes with stationary increments. Removing this jump from Z, and calling the remaining process again Z, we write this case of Eq. (5) as

$$Y(t) = Y(0) + Z_0 t + \int_{-\infty}^{\infty} \frac{\exp(i\omega t) - 1}{i\omega}\, dZ(\omega) \tag{7}$$

where $E\,|\,dZ(\omega)|^2 = f(\omega)\,d\omega$, f being a nonnegative function such that

$$\int_{-\infty}^{\infty} \frac{f(\omega)}{1+\omega^2}\, d\omega < \infty \tag{8}$$

In this situation we will say that Y has the *formal spectral density* $f(\omega)/\omega^2$. The point is that maybe

$$\int_{-1}^{1} \frac{f(\omega)}{\omega^2}\, d\omega = \infty \tag{9}$$

in which case $f(\omega)/\omega^2$ cannot be the spectral density of any stationary process. The main purpose of this article is to make some sense of this terminology.

Now it is evident what our model for flicker noise shall be, namely, a process Y with stationary increments having a formal spectral density $1/|\,\omega\,|$. This would make $f(\omega) = |\,\omega\,|$, which violates Eq. (8). Accordingly, we demand a $1/|\,\omega\,|$ behavior only for low frequencies. We will require that

$$\frac{f(\omega)}{|\,\omega\,|} \rightarrow 1 \qquad \text{as } \omega \rightarrow 0 \tag{10}$$

and that f roll off enough at high frequencies to satisfy Eq. (8). The exact nature of the rolloff will not affect our results.

III. Quadratic Means

We wish to know how these processes behave under certain measurements. Let X be a stationary process with spectral density f. We will consider only measurements of the following form:

$$P = \left| \int_{-\infty}^{\infty} X(t)h(t)dt \right|^2 \tag{11}$$

where h is a complex-valued "time window" such that

$$\int_{-\infty}^{\infty} |\,h(t)\,|\, dt < \infty \tag{12}$$

(We may include δ-functions in h.) Further, we will look only at the expectation of P, ignoring the problem of finding its variance under assumptions about higher moments of X. It is a familiar fact that

$$EP = \int_{-\infty}^{\infty} |\,H(\omega)\,|^2 f(\omega) d\omega \tag{13}$$

where

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \exp(i\omega t) dt \tag{14}$$

Now suppose that we make the same measurement on a process Y with stationary increments and a formal spectral density $f(\omega)/\omega^2$. Assume $Y(0) = 0$ for now. Let

$$Q = \left| \int_{-\infty}^{\infty} Y(t)h(t)dt \right|^2 \tag{15}$$

The integral can be transformed as follows:

$$\int_{-\infty}^{\infty} Y(t)h(t)dt - Z_0\int_{-\infty}^{\infty} th(t)dt$$
$$= \int_{-\infty}^{\infty} dth(t) \int_{-\infty}^{\infty} dZ(\omega)\frac{\exp(i\omega t) - 1}{i\omega}$$
$$= \int_{-\infty}^{\infty} dZ(\omega)\frac{1}{i\omega}\int_{-\infty}^{\infty} dth(t)[\exp(i\omega t) - 1]$$
$$= \int_{-\infty}^{\infty} \frac{H(\omega) - H(0)}{i\omega} dZ(\omega) \tag{16}$$

The condition

$$\int_{-\infty}^{\infty} (1 + |t|)|h(t)|dt < \infty \tag{17}$$

plus Eq. (8), is sufficient to validate the interchange of orders of integration leading to Eq. (16). Condition (17) also makes H differentiable. From Eqs. (16), (4), and the orthogonality of Z_0 to dZ, we get

$$EQ = |H'(0)|^2 E|Z_0|^2 + \int_{-\infty}^{\infty} |H(\omega) - H(0)|^2 \frac{f(\omega)}{\omega^2} d\omega \tag{18}$$

valid when $Y(0) = 0$ (or, what is the same, if h is applied to $Y(t) - Y(0)$). The formal spectral density $f(\omega)/\omega^2$ appears in somewhat the same role as spectral density $f(\omega)$ does in Eq. (13), the analogous formula for stationary processes. The next two sections examine some special cases of these measurements.

IV. High-Pass Filters

We say that a time window h satisfying Eq. (17) is a high-pass filter if $H(0) = 0$, i.e.,

$$\int_{-\infty}^{\infty} h(t)dt = 0 \tag{19}$$

In this situation, we no longer need to require $Y(0) = 0$, and Eq. (18) becomes

$$EQ = |H'(0)|^2 E|Z_0|^2 + \int_{-\infty}^{\infty} |H(\omega)|^2 \frac{f(\omega)}{\omega^2} d\omega \tag{20}$$

The first term of Eq. (20) is due to linear drift. The second term is exactly what we would get if a stationary process with spectral density $f(\omega)/\omega^2$ were subjected to the same measurement. This is what the first approach (1) of Section I gives. Of course, if Eq. (9) holds, there is no such stationary process. Moreover, if h is not high-pass, then approach (1) fails, for the integral in Eq. (20) diverges.

In the study of oscillator stability, $Y(t)$ is the frequency of an oscillator at time t, relative to some nominal average frequency. To measure the instability of Y, one often uses a family of high-pass filters

$$h_\tau(t) = \frac{1}{\tau}k\left(\frac{t}{\tau}\right)$$

depending on an integration-time parameter τ. Here, $k(x)$ is a fixed high-pass filter function of dimensionless time x. Perhaps the simplest of these is given by

$$\begin{aligned} k(x) &= -1/2^{1/2}, & 0 < x < 1 \\ &= 1/2^{1/2}, & 1 < x < 2 \\ &= 0, & \text{otherwise} \end{aligned}$$

The corresponding Q is called the Allan variance. It is the sample variance of two successive averages over adjacent time intervals. Let

$$K(y) = \int_{-\infty}^{\infty} \exp(ixy)k(x)dx$$

Then

$$EQ = \tau^2|K'(0)|^2 E|Z_0|^2 + \int_{-\infty}^{\infty} |K(\omega\tau)|^2 \frac{f(\omega)}{\omega^2} d\omega \tag{21}$$

For the flicker noise case, let us assume for simplicity that $f(\omega) = |\omega|$ for $|\omega| \leq \omega_1$. (Actually, conditions (8) and (10) are sufficient.) The integral in Eq. (21) becomes

$$\int_{-\omega_1\tau}^{\omega_1\tau} |K(y)|^2 \frac{dy}{|y|} + \int_{|\omega|>\omega_1} |K(\omega\tau)|^2 \frac{f(\omega)}{\omega^2} d\omega$$

If $K(y)$ tends to 0 fast enough as $y \to \infty$, this expression tends to

$$\int_{-\infty}^{\infty} |K(y)|^2 \frac{dy}{|y|} \tag{22}$$

as $\tau \to \infty$. In the case of Allan variance, this integral is

$$\int_0^{\infty} \sin^4 \frac{1}{2}y \frac{1}{y^3} dy = 4 \log 2$$

A standard test for the presence of flicker noise is the leveling off of the Allan variance to a nonzero limit as τ gets large. Since $K'(0) \neq 0$ in this case, the linear drift term in Eq. (21) grows like τ^2. Thus it is obviously necessary to remove linear drift from Y before making the measurement, and this is in fact done (Ref. 5). Of course, this surgery cannot be performed without damaging the second term of Eq. (21); we will examine this situation in detail in Section VI. This difficulty can be avoided by using a filter h such that $H'(0) = 0$, i.e.,

$$\int_{-\infty}^{\infty} th(t)dt = 0 \tag{23}$$

Then the linear drift term vanishes. When Barnes (Ref. 4) considers the third difference of the phase of an oscillator, he is using such a filter.

V. Spectral Estimates

We wish to see what happens when we perform experiments on our model that are designed to measure spectral density f of a stationary process X. One estimate of f at a chosen frequency ω_0 is a modified periodogram (Refs. 15 and 16). Let k be an integrable function on (0,1). The estimate of $f(\omega_0)$ using an integration time of τ is

$$I_\tau(\omega_0) = \frac{1}{2\pi\tau}\left| \int_0^\tau X(t)k\left(\frac{t}{\tau}\right) \exp(-i\omega_0 t)\, dt \right|^2 \tag{24}$$

To get stable estimates of $f(\omega_0)$ we would have to average $I_\tau(\omega)$ over a band of frequencies that is wide compared with $1/\tau$. We will not do this here.

Let

$$K(y) = \int_0^1 \exp(ixy)k(x)dx \tag{25}$$

The measurement $I_\tau(\omega_0)$ is of the Eq. (11) type. By Eq. (13),

$$EI_\tau(\omega_0) = \frac{\tau}{2\pi}\int_{-\infty}^{\infty} |K((\omega - \omega_0)\tau)|^2 f(\omega)d\omega \tag{26}$$

Assume that k is square-integrable, and that f is a bounded, continuous function on $(-\infty, \infty)$. Then

$$EI_\tau(\omega_0) \rightarrow f(\omega_0)\int_0^1 |k(x)|^2 dx$$

as $\tau \rightarrow \infty$; in other words, $I_\tau(\omega_0)$ is an asymptotically unbiased estimate of $f(\omega_0) \int |k|^2$.

Let $J_\tau(\omega_0)$ be defined as in Eq. (24), except that X is replaced by a process Y with stationary increments and a formal spectral density $f(\omega)/\omega^2$. Assume again that $Y(0) = 0$. By Eq. (18),

$$EJ_\tau(\omega_0) = \frac{1}{2\pi}\tau^3 |K'(-\omega_0\tau)|^2 E|Z_0|^2 + \frac{\tau}{2\pi}\int_{-\infty}^{\infty} |K((\omega - \omega_0)\tau) - K(-\omega_0\tau)|^2 \frac{f(\omega)}{\omega^2}d\omega \tag{27}$$

Suppose that $K(y)$ and $K'(y)$ tend to 0 faster than $|y|^{-3/2}$ as $|y| \rightarrow \infty$. As $\tau \rightarrow \infty$, the linear drift term goes to 0. In the integral, the $1/\omega^2$ divergence is cancelled by the $|\cdots|^2$ factor. For ω near ω_0, the term $K(-\omega_0\tau)$ is insignificant compared to $K((\omega - \omega_0)\tau)$. As a result, when $\tau \rightarrow \infty$, expression (27) behaves like (26), this time picking off the value $f(\omega)/\omega_0^2$ of the formal spectral density. For flicker noise we set $f(\omega) \sim |\omega|$ as $\omega \rightarrow 0$, and we see again how the model manages to masquerade as a nonexistent stationary process with spectral density $\sim 1/|\omega|$ for small ω.

Here is a precise statement about the behavior of $EJ_\tau(\omega_0)$: *Let k be an absolutely continuous function on* [0,1] *such that $k(0) = k(1) = 0$, and let K be its Fourier transform Eq. (25). Let f be a continuous function satisfying Eq. (8). Then for $\omega_0 \neq 0$, the second term of Eq. (27) tends to*

$$\frac{f(\omega_0)}{\omega_0^2}\int_0^1 |k(x)|^2 dx \tag{28}$$

as $\tau \rightarrow \infty$.

Proof: The conditions on k imply

$$K^{(n)}(y) = o\left(\frac{1}{|y|}\right), \qquad \text{as } |y| \rightarrow \infty \tag{29}$$

for $n = 0, 1, 2, \cdots$.

It will be enough to prove the result when $\omega_0 > 0$. We break up the integral in Eq. (27) as follows:

$$\frac{\tau}{2\pi}\left(\int_{-\infty}^{-1/\tau} + \int_{-1/\tau}^{1/\tau} + \int_{1/\tau}^{\omega_0/2} + \int_{\omega_0/2}^{3\omega_0/2} + \int_{3\omega_0/2}^{\infty}\right)$$
$$= I_1 + I_2 + I_3 + I_4 + I_5$$

We will show that I_4 tends to Eq. (28) as $\tau \to \infty$, and the other I_j tend to 0. Let $q = \omega_0 \tau$, and let

$$g(x) = \frac{f(\omega_0(1+x))}{\omega_0^2(1+x)^2}$$

Then

$$I_4 = \int_{-q/2}^{q/2} |K(y) - K(-q)|^2 g\left(\frac{y}{q}\right) \frac{dy}{2\pi}$$

$$= \int_{-m}^{m} + \int_{m<|y|<q/2} \qquad (30)$$

As $q \to \infty$, the first integral in Eq. (30) tends to

$$\frac{g(0)}{2\pi} \int_{-m}^{m} |K(y)|^2 dy \qquad (31)$$

The second integral in Eq. (30) is less than

$$\frac{1}{\pi} max\left\{g(x) : |x| < \frac{1}{2}\right\}\left(\int_{|y|>m} |K(y)|^2 dy + q|K(-q)|^2\right)$$

By choosing m large we can make this expression as small as we like for all q sufficiently large, and can also make Eq. (31) as close as we like to

$$\frac{g(0)}{2\pi} \int_{-\infty}^{\infty} |K(y)|^2 dy = \frac{f(\omega_0)}{\omega_0^2} \int_0^1 |K(x)|^2 dx$$

This establishes the limiting behavior of I_4.

To estimate I_1 we simply observe

$$\tau |K((\omega - \omega_0)\tau) - K(-\omega_0\tau)|^2 = o\left(\frac{1}{\tau}\right)$$

as $\tau \to \infty$, uniformly for $\omega < 0$. Hence

$$I_1 \leq o\left(\frac{1}{\tau}\right) \int_{-\infty}^{-1/\tau} \frac{f(\omega)}{\omega^2} d\omega$$

$$= o\left(\frac{1}{\tau}\right) O(\tau) = o(1)$$

The estimates for I_3 and I_5 are similar.

Only I_2 remains. By a version of the mean value theorem for complex-valued functions of a real variable,

$$|K(\omega\tau - \omega_0\tau) - K(-\omega_0\tau)|^2 \leq \omega^2\tau^2 |K'(c)|^2$$

for some number c between $\omega\tau - \omega_0\tau$ and $-\omega_0\tau$. If $|\omega| \leq 1/\tau$ and $\tau \geq 2/\omega_0$, then $c \leq -\omega_0\tau/2$ and $|K'(c)|^2 = o(1/\tau^2)$. Therefore

$$I_2 \leq o(1)\, \tau \int_{-1/\tau}^{1/\tau} f(\omega) d\omega = o(1)$$

The proof is complete.

When $f(0) = 0$, as in the flicker noise model, we can replace o by O in Eq. (29). Then even a boxcar function will serve for k. Of course, to make the linear drift term in Eq. (27) tend to 0 we need $K'(y) = o(1/|y|)^{3/2}$. This problem goes away when we remove drift before doing the spectral analysis; we treat this situation in the next section.

VI. Removal of Linear Trends

Before taking the kinds of measurements we have described, it is common practice to fit a linear trend to the data and subtract it off. Measurements are then taken on the residual data. In spectral measurements this avoids interaction of dc components with minor lobes of the spectral window.

Least-squares fitting on an interval $-1/2\tau \leq t \leq 1/2\tau$ is convenient for us here. Given a signal $u(t)$, we produce a residual signal

$$u_r(t) = u(t) - a_0 - a_1 t$$

where

$$a_0 = \frac{1}{\tau} \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} u(t) dt, \qquad a_1 = \frac{12}{\tau^3} \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} t u(t) dt$$

Then

$$\int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} u_r(t) dt = 0, \qquad \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} t u_r(t) dt = 0 \qquad (32)$$

and for any two given signals u and v,

$$\int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} u_r(t) v(t) dt = \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} u(t) v_r(t) dt \qquad (33)$$

Let Y be the process of Eq. (7). We will look at quadratic means Q_r of the reduced process $Y_r(t)$:

$$Q_r = \left| \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} Y_r(t) h(t) dt \right|^2$$

If h is integrable on $[-\tau/2, \tau/2]$,

$$\int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} Y_r(t)h(t)dt = \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} Y(t)h_r(t)dt \qquad (34)$$

Let

$$H(\omega) = \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} h(t) \exp(i\omega t)dt$$

$$H_r(\omega) = \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} h_r(t) \exp(i\omega t)dt \qquad (35)$$

Because of Eq. (32), not only is h_r a high-pass filter, $H_r(0) = 0$, but also $H_r'(0) = 0$. Hence, by Eq. (20),

$$EQ_r = \int_{-\infty}^{\infty} |H_r(\omega)|^2 \frac{f(\omega)}{\omega^2} d\omega \qquad (36)$$

Take the situation

$$h(t) = \frac{1}{\tau} k\left(\frac{t}{\tau}\right)$$

where k is a function with residual k_r defined with respect to the interval $[-1/2, 1/2]$. Then

$$h_r(t) = \frac{1}{\tau} k_r\left(\frac{t}{\tau}\right)$$

and Eq. (21) gives

$$EQ_r = \int_{-\infty}^{\infty} |K_r(\omega\tau)|^2 \frac{f(\omega)}{\omega^2} d\omega \qquad (37)$$

where

$$K_r(y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} k_r(x) \exp(ixy)dx$$

For the flicker noise case, $f(\omega) \sim |\omega|$ as $\omega \rightarrow 0$, EQ_r tends to

$$\int_{-\infty}^{\infty} |K_r(y)|^2 \frac{dy}{|y|}$$

as $\tau \rightarrow \infty$, and there is no linear drift term to interfere with our observations. This happens whether or not k is a high-pass filter.

Before we look at spectral measurements on the reduced process Y_r, we need a formula for H_r in terms of H. By Eqs. (35) and (33),

$$H_r(\omega) = \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} h(t)e(\omega,t)\, dt$$

where $e(\omega,t)$ is the reduced form of the function $\exp(i\omega t)$ on $-\tau/2 \leq t \leq \tau/2$. We calculate

$$e(\omega,t) = \exp(i\omega t) - \phi_0(\omega\tau/2) - \phi_1(\omega\tau/2)i\omega t$$

where

$$\phi_0(x) = \frac{\sin x}{x}, \qquad \phi_1(x) = 3\frac{\sin x - x\cos x}{x^3}$$

Therefore the desired formula is

$$H_r(\omega) = H(\omega) - \phi_0(\omega\tau/2)H(0) - \phi_1(\omega\tau/2)\omega H'(0) \qquad (38)$$

We apply Eqs. (36) and (38) to the spectral estimate $J_\tau(\omega_0)$ of Section V, where Y is replaced by Y_r, and k is on $[-1/2, 1/2]$ instead of $[0,1]$. We get

$$EJ_\tau(\omega_0) = \frac{\tau}{2\pi}\int_{-\infty}^{\infty} |K((\omega-\omega_0)\tau) - \phi_0(\omega\tau/2)K(-\omega_0\tau) - \phi_1(\omega\tau/2)\omega\tau K'(-\omega_0\tau)|^2 \frac{f(\omega)}{\omega^2}d\omega \qquad (39)$$

Although this is messier than Eq. (27), it is actually better behaved. There is no linear drift term. When $|\omega\tau| \geq 1$, the perturbing terms in (39) go to 0 at least as fast as the old $K(-\omega_0\tau)$, and even faster when ω is bounded away from 0. In the region $|\omega\tau| \leq 1$, the integrand of (39) is like that of (27) except for the extra terms

$$(1 - \phi_0(\omega\tau/2))K(-\omega_0\tau) - \phi_1(\omega\tau/2)\omega\tau K'(-\omega_0\tau)$$

which are both $|\omega| o(1)$ as $\tau \rightarrow \infty$, $|\omega\tau| \leq 1$. Hence this part of the integral behaves as well as before.

We conclude that the reduced $EJ_\tau(\omega_0)$ tends to the right side of Eq. (28). In general, the detrending operation enhances the ability of our measurements to provide information about the formal spectral density $f(\omega)/\omega^2$ of a process with stationary increments.

VII. Approximation by Stationary Processes

The purpose of this section is to give a concrete interpretation of the method of cutoffs, the second approach of Section I. Rather than creating out of nothing a stationary process with spectral density that cuts off below $\omega = \epsilon$, we will generate one by sending the process Y of Eq. (7) through a high-pass filter. This will give a stationary process X_ϵ which in a certain sense converges to Y as $\epsilon \to 0$.

We will need to assume that the linear drift term $Z_0 t$ is absent from $Y(t)$. Take a high-pass filter with impulse response

$$h_\epsilon(t) = \delta(t) - \epsilon k(\epsilon t)$$

where $\epsilon > 0$ and k is a function such that

$$\int_{-\infty}^{\infty} (1 + |x|)\,|k(x)|\,dx < \infty, \int_{-\infty}^{\infty} k(x)dx = 1$$

and δ is the Dirac delta. This time, let

$$K(y) = \int_{-\infty}^{\infty} \exp(-ixy)\,k(x)dx$$

a bounded, differentiable function that tends to 0 as $|y| \to \infty$. Define the process X_ϵ by

$$X_\epsilon(t) = \int_{-\infty}^{\infty} Y(s)h_\epsilon(t-s)ds$$

Then Eq. (16) gives

$$X_\epsilon(t) = \int_{-\infty}^{\infty} \exp(i\omega t)\frac{1 - K(\omega/\epsilon)}{i\omega}\,dZ(\omega)$$

Hence, X_ϵ is a stationary process with spectral density

$$\left|1 - K\left(\frac{\omega}{\epsilon}\right)\right|^2 \frac{f(\omega)}{\omega^2} \tag{40}$$

a cutoff version of $f(\omega)/\omega^2$ which tends to $f(\omega)/\omega^2$ as $\epsilon \to 0$. If Eq. (9) holds, the random variables $X_\epsilon(t)$, t fixed, $\epsilon \to 0$, do not converge to anything, since $E\,|X_\epsilon(t)|^2 \to \infty$ as $\epsilon \to 0$. Nevertheless, for each t we do have

$$X_\epsilon(t) - X_\epsilon(0) \to Y(t) - Y(0)$$

in mean-square, as $\epsilon \to 0$.

Proof: Let $\Delta Y(t) = Y(t) - Y(0)$, and similarly for X_ϵ. Then

$$\Delta X_\epsilon = \int_{-\infty}^{\infty} \frac{\exp(i\omega t) - 1}{i\omega}\left(1 - K\left(\frac{\omega}{\epsilon}\right)\right) dZ(\omega) \tag{41}$$

Equations (41), (7), and (4) give

$$E\,|\Delta Y(t) - \Delta X_\epsilon(t)|^2 = \int_{-\infty}^{\infty}\left(\frac{\sin \omega t/2}{\omega/2}\right)^2 \left| - K\left(\frac{\omega}{\epsilon}\right)\right|^2 f(\omega)d\omega$$

Since

$$\left(\frac{\sin \omega t/2}{\omega/2}\right)^2 (1 + \omega^2) \leq t^2 + 4$$

we have

$$E\,|\Delta Y(t) - \Delta X_\epsilon(t)|^2 \leq (t^2 + 4)\int_{-\infty}^{\infty}\left| - K\left(\frac{\omega}{\epsilon}\right)\right|^2 \frac{f(\omega)}{1+\omega^2}\,d\omega \tag{42}$$

which, by Lebesgue's dominated convergence theorem, tends to 0 as $\epsilon \to 0$.

If h is a time window satisfying Eq. (17), then (42) implies

$$\int_{-\infty}^{\infty} \Delta X_\epsilon(t)h(t)dt \to \int_{-\infty}^{\infty} \Delta Y(t)h(t)dt \tag{43}$$

in mean-square, as $\epsilon \to 0$. If h is also a high-pass filter, then

$$\int_{-\infty}^{\infty} X_\epsilon(t)h(t)dt \to \int_{-\infty}^{\infty} Y(t)h(t)dt$$

so that in this case, the method of cutoffs fits smoothly into our model.

Strictly speaking, we have not expressed $\Delta Y(t)$ as a limit of stationary processes, but rather as the limit of $\Delta X_\epsilon(t)$, which is not stationary but has stationary increments [and a *formal* spectral density (Eq. 40)]. The stationary process X_ϵ, ϵ very small, would not be a good model for flicker noise because in this case $E\,|X_\epsilon(t)|^2$ is large for all t. An appropriate model might be ΔX_ϵ, but then we might as well use Y, which does not have an extra parameter ϵ to make calculations messier.

VIII. The Barnes–Allan Model

In 1966, Barnes and Allan (Ref. 17) exhibited a flicker noise model and calculated its Allan variance. (See Section IV for a definition.) We have examined the behavior of our own model under a general class of measurements, and will now do the same for the Barnes–Allan model. The details of the derivations will be omitted.

Their model for the frequency $\Phi'(t)$ of an oscillator (Φ is the phase) is given by

$$\Phi'(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(t-u)^{1/2}} \, dW(u), \qquad t \geq 0 \tag{44}$$

where dW is white noise. Actually, this stochastic integral does not exist, since

$$\int_0^t \frac{du}{t-u} = \infty$$

Nevertheless, Eq. (44) defines a *generalized* process; that is, we can give a meaning to

$$\int_0^\infty \Phi'(t) h(t) dt \tag{45}$$

for suitable time windows h by formally plugging Eq. (44) into (45) and reversing orders of integration. (The real reason that Φ is not an ordinary process is that there is no high-frequency cutoff of the $1/|\omega|$ spectral behavior.)

From here on, let h be of bounded variation and equal to 0 outside some interval $[0,b]$. Let R be the square of the absolute value of Eq. (45). We are able to show that

$$ER = \frac{1}{2\pi} \int_{-\infty}^\infty |H(\omega) - H^*(\omega)|^2 \frac{d\omega}{|\omega|} \tag{46}$$

where H is given by (14) and

$$H^*(\omega) = \int_0^\infty H(\omega\xi) p(\xi) d\xi \tag{47}$$

$$p(\xi) = \frac{1}{\pi(1+\xi)\xi^{1/2}}$$

Since $\int p = 1$, $H^*(\omega)$ is a weighted average of H. Formula (46) is analogous to (18). In our own model Y, we set $Z_0 = 0$ and $f(\omega) = |\omega| g(\omega)$, where $g(\omega) \to 1$ as $\omega \to 0$. Then (18) becomes

$$EQ = \int_{-\infty}^\infty |H(\omega) - H(0)|^2 g(\omega) \frac{d\omega}{|\omega|} \tag{48}$$

Since Y has stationary increments, the starting time T of measurements does not matter, provided we replace $Y(t) - Y(0)$ by $Y(T+t) - Y(T)$. This is not the case for Φ', as Barnes and Allan recognize. Therefore we first consider h of form

$$h(t) = \frac{1}{\tau} k\left(\frac{t-T}{\tau}\right)$$

where k is not necessarily high-pass. If we set $Y(T) = 0$, then EQ depends only on τ, namely,

$$EQ = \int_{-\infty}^\infty |K(y) - K(0)|^2 g\left(\frac{y}{\tau}\right) \frac{dy}{|y|} \tag{49}$$

and

$$EQ \to \int_{-\infty}^\infty |K(y) - K(0)|^2 \frac{dy}{|y|} \qquad \text{as } \tau \to \infty \tag{50}$$

On the other hand, ER depends only on $\rho = T/\tau$, and we can show that

$$ER \to \frac{1}{2\pi} \int_{-\infty}^\infty |K(y) - K^*(y)|^2 \frac{dy}{|y|} \qquad \text{as } \rho \to 0 \tag{51}$$

$$ER \to \frac{1}{2\pi} \int_{-\infty}^\infty |K(y)|^2 \frac{dy}{|y|} \qquad \text{as } \rho \to \infty \tag{52}$$

If K is high-pass, $K(0) = 0$, then all these limits are finite. For the special case of Allan variance, Barnes and Allan keep τ fixed and let $T \to \infty$ since (Ref. 17) "flicker noise is normally observed on equipment which has been operating for long periods of time." From Eqs. (52) and (50), we see that $2\pi \lim_{T\to\infty} ER = \lim_{T\to\infty} EQ$. Thus, $2\pi ER$ and EQ are almost equal for large τ and $T >> \tau$. If we keep T fixed and let $\tau \to \infty$, then EQ approaches the right side of (50), whereas ER approaches the right side of (51).

If K is not high-pass, $K(0) \neq 0$, then the right sides of (50) and (52) are infinite, whereas (51) is still finite. If τ is large and $T >> \tau$, then EQ and ER are both large. If T is fixed and $\tau \to \infty$, then $EQ \to \infty$, while ER remains finite. Because of the ambiguity of T, the Barnes–Allan model cannot be used to predict the dependence of R on τ if $K(0) \neq 0$, while (49) does exactly that for Q. In fact, it is easy to see that EQ grows like $\log \tau$.

We have also calculated the expected modified periodogram of Φ' for the case $T = 0$. This is the expectation of the expression (24) with X replaced by Φ'. If k is of bounded variation on $[0,1]$, an effort as in Section V shows that the expected modified periodogram tends to

$$\frac{1}{2\pi|\omega_0|} \int_0^1 |k(x)|^2 dx$$

as $|\omega_0|\tau \rightarrow \infty$. Thus, the Barnes–Allan model does display a $1/|\omega|$ behavior when subjected to a spectral measurement.

The two models Y and Φ' are closely related. Formulas (46) and (48) display the relationship in the spectral domain. It is possible to generate a formula for $Y(t)$ in the time domain by passing white noise through a certain realizable filter, then integrating. The resulting expression splits naturally into two parts, one of which resembles (44). This part is actually a version of the Barnes–Allan model that cuts off the $1/|\omega|$ behavior at high frequencies.

IX. Future Prospects

We see two directions for further work. First of all, we would like to make more comparisons of the behavior of actual flicker noise with the behavior of our model Y and the Barnes–Allan model Φ'. These models already agree with the experiments involving high-pass filter averaging and direct measurements of spectral density, but Φ' has some difficulty predicting the result of non-high-pass filter averages. Averaging experiments could be done on flicker noise data to search for the logarithmic dependence on integration time that Eq. (49) predicts for Y.

Secondly, it would be desirable to search for physical mechanisms that could generate either model Y or Φ'. Since we have described only second-moment properties of these models, each model can be realized in a variety of ways. Instead of using Brownian motion to generate the model, we can start with other processes with orthogonal increments. For example, if we started with a suitably modified Poisson process, we would get some form of nonstationary shot noise. Such a noise might occur in the frequency of an oscillator subject to infrequent but sudden random disturbances.

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